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The Fourth Indeterminacy Relation and Stochastic Electrodynamics

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Abstract

An interesting conservation property and the time-energy uncertainty are derived in the framework of stochastic electrodynamics.

1. *Introduction*

Stochastic electrodynamics is an attempt, using exclusively classical arguments, to derive quantum-mechanical results.

The development of stochastic electrodynamics is based on classical physics, special relativity and on the concept of a universal fluctuating electromagnetic field at absolute zero of temperature (the zero-point field).

The application of the Lorenz-invariance to the energy spectrum of the zero-point field yields for the one-dimensional case the following expression of the spectral density:

$$E_\delta(\omega) = \frac{K\omega^3}{3\pi c^3} \tag{1.1}$$

where K is a constant, having the dimension of action.

Using this unique information about the zero-point field and considering that its action is Markoffian, numerous quantum-like results were obtained:

the relation giving the ground state energy of a harmonic oscillator;
the relation giving the energy of a free electron moving in a uniform magnetic field, involving the 'spin' of the electron;
Planck's law of black-body radiation;
the probability distribution laws of x and of $\dot{x} = p/m$, for the harmonic oscillator and for a free electron moving in a uniform magnetic field.
in the above two cases, the uncertainty relation

$$\Delta x . \Delta p = K \tag{1.2}$$

Schrödinger's equation, with two extra terms.

To achieve complete identification of the ensemble of the results obtained in stochastic electrodynamics with those of quantum mechanics, one has to show that:

$$K = \hbar$$

Such a demonstration and a fuller account of the preceding may be found in Surdin (1971).

It appeared interesting to complement (1.2) by considering within the framework of stochastic electrodynamics the time-energy indeterminacy.

These considerations were also prompted by the fact that some discord appeared regarding the time-energy indeterminacy in quantum mechanics.

The usual derivation of the fourth, time-energy indeterminacy relation

$$\Delta_\psi t . \Delta_\psi E \geqslant \hbar \tag{1.3}$$

where $\Delta_\psi t$ and $\Delta_\psi E$ are the mean standard deviations of time and energy for a quantum-mechanical system in a state Ψ, is obtained by considering a wave packet. If Δx is the linear extension of the wave packet, v its group velocity, the uncertainty in the determination of the time instant when the packet passes at a given point in space is

$$\Delta t \simeq \frac{\Delta x}{v}$$

The uncertainty in the determination of the energy is

$$\Delta E \simeq \frac{\partial E}{\partial p} \Delta p = v . \Delta p$$

hence

$$\Delta t . \Delta E \simeq \Delta x . \Delta p$$

using (1.2), one obtains (1.3).

This type of derivation was strongly contested by Bunge (1970) mainly on the following grounds:

relation (1.3), unlike the genuine relation (1.2), has never been proved from first principles;

whereas the energy is a dynamical variable, t, being a parameter, is not.

Thus for every Ψ one must have $\Delta_\psi t = 0$. Moreover, in cases where the system happens to be in an eigenstate of the energy operator, the standard deviation of the energy vanishes as well.

In conclusion Bunge proposed that formula (1.3) be dropped altogether.

Allcock (1969) has carefully analysed formula (1.3) and other formulae reminiscent of it; he arrives, essentially, to a similar, although not as drastic, conclusion as Bunge.

Several more sophisticated derivations of (1.3) were advanced, e.g. Fujiwara (1970) and Durand (1970). The latter proposes to consider the time average of an operator time; t then becomes a random variable as do the space variables. The time average of t, $\langle\langle t \rangle\rangle = t_0$ has then a physical

meaning: it is the instant when the probability $|\Psi(r_0, t)|^2$, at a fixed point in space r_0, has its maximum value. With the help of the usual quantum-mechanical formalism and the above considerations Durand derives (1.3) in the case of continuous energy spectra.

The above brief summary of opposing opinions leads one to consider that in quantum mechanics (1.3) is shrouded with some uncertainty and that more clarification is desirable.

A possible contribution to the desired clarification might come from considerations of the time-energy uncertainty in stochastic electrodynamics. However, before dealing with the time-energy uncertainty, an interesting conservation property will be presented.

2. *A Conservation Property*

Consider the one-dimensional motion of a particle of mass m and electric charge e, moving in the zero-point field and in a field $F = -(dU/dx)$, where the potential $U(x)$ is finite for all x.

The equation of motion in the non-relativistic case† is:

$$-\tau\dddot{x} + \ddot{x} + \frac{1}{m}\frac{dU}{dx} = \frac{e}{m}\mathscr{E}(t) \tag{2.1}$$

where $\tau = 1/\omega_s = 2e^2/3mc^3$ and $\mathscr{E}(t)$ is the fluctuating zero-point electric field.

Multiplication of both sides of (2.1) by $\dot{x}$, gives

$$-\tau\ddot{x}.\dot{x} + \ddot{x}.\dot{x} + \frac{1}{m}\frac{dU}{dx}\cdot\dot{x} = \frac{e}{m}\dot{x}.\mathscr{E}(t) \tag{2.2}$$

An equivalent form of (2.2) is:

$$\tau\ddot{x}^2 - \tau\frac{d}{dt}(\ddot{x}.\dot{x}) + \frac{1}{2}\frac{d}{dt}(\dot{x}^2) + \frac{1}{m}\frac{d}{dt}v = \frac{e}{m}\dot{x}\mathscr{E}(t) \tag{2.3}$$

The time average of both sides of (2.3) yields

$$\tau\langle\langle\ddot{x}^2\rangle\rangle - \tau\frac{d}{dt}\langle\langle\ddot{x}.\dot{x}\rangle\rangle + \frac{1}{2}\frac{d}{dt}\langle\langle\dot{x}^2\rangle\rangle + \frac{1}{m}\frac{d}{dt}\langle\langle U\rangle\rangle = \frac{e}{m}\langle\langle\dot{x}.\mathscr{E}\rangle\rangle \tag{2.4}$$

The zero-point field, x, and its time derivatives are centred stationary random variables, their time averages are independent of the time t. Hence

$$\tau\langle\langle\ddot{x}^2\rangle\rangle = \frac{e}{m}\langle\langle\dot{x}\mathscr{E}\rangle\rangle \tag{2.5}$$

This equation is interpreted as follows: the average energy $m.\tau.\langle\langle\ddot{x}^2\rangle\rangle$, emitted per unit time by the particle is exactly compensated by the average

† In the non-relativistic case the term $e.\mathbf{v}\wedge\mathbf{B}(t)$ is neglected. If, however, this term is retained one may still regain three relations similar to (2.5), one for each coordinate axis, since x, y, and z, as well as $E_x(t)$, $E_y(t)$ and $E_z(t)$ are considered to be independent random variables.

energy $e.\langle\langle \dot{x}.\mathscr{E}\rangle\rangle$, conceded by the zero-point field to the particle during the same time.

This is also the necessary condition for the existence of a stationary state. Thus, for a harmonic oscillator, where $U(x) = 1/2kx^2$, Braffort & Tzara (1954) and Braffort *et al.* (1965) have shown that a stationary state exists and that the energy of the ground state is

$$E_0 = \frac{K\omega_0}{2}\left[1 - \frac{1}{2\pi}\frac{\omega_0}{\omega_s}\log\frac{\omega_0}{\omega_s}\right] \tag{2.6}$$

where $\omega_0^2 = k/m$.

3. *Time-Energy Uncertainty in Stochastic Electrodynamics*

The simple case of a non-relativistic harmonic oscillator is considered. The arguments used apply also to the case of a free electron moving in a uniform magnetic field.

The equation of motion of the harmonic oscillator is

$$-\tau\dddot{x} + \ddot{x} + \omega_0^2 x = \frac{e}{m}\mathscr{E}(t) \tag{3.1}$$

For the non-relativistic case $\tau\omega_0 \ll 1$; (3.1) can be approximated by a generalised Langevin equation (Surdin, 1970)

$$\ddot{x} + \tau\omega_0^2\dot{x} + \omega_0^2 x = \frac{e}{m}\mathscr{E}(t) \tag{3.2}$$

Considering that the action of the fluctuating zero-point field on the oscillator is a Markoffian process, the steady-state probability distribution $p(x,\dot{x})$ is given by (Surdin, 1970)

$$p(x,\dot{x}) = C^t \exp\left[-\frac{m(\dot{x}^2 + \omega_0^2 x^2)}{K\omega_0\left(1 - \frac{1}{\pi}\frac{\omega_0}{\omega_s}\log\frac{\omega_0}{\omega_s}\right)}\right] \tag{3.3}$$

to obtain (3.3), (1.1) was used.

One deals here with the case of a very lightly damped oscillator moving in a random force field. Due to its very low damping the oscillator possesses a very narrow pass-band, so that it responds, essentially, to the ω_0 component of the fluctuating field.

Under these conditions the energy of the oscillator, to a good approximation, is

$$E = \frac{m}{2}(\dot{x}^2 + \omega_0^2 x^2) \tag{3.4}$$

Using (3.3), the probability distribution $p(E)$ of the energy is

$$p(E) = C^t.E.\exp\left(-\frac{2E}{K\omega_0}\right) \tag{3.5}$$

where only the main term in the denominator of the exponential in (3.3) is retained.

The ensemble average value $\langle E \rangle$, deduced from (3.5), is

$$\langle E \rangle = E_0 = \tfrac{1}{2} K \omega_0 \tag{3.6}$$

Neglecting terms of the first order in ω_0/ω_s and higher orders, the ensemble average (3.6) is the same as the time average (2.6).

The mean square value $\langle E^2 \rangle$ is:

$$\langle E^2 \rangle = 2E_0{}^2 \tag{3.7}$$

and the mean standard deviation

$$\Delta E = [\langle E^2 \rangle - \langle E \rangle^2]^{1/2} = \frac{K\omega_0}{2} = E_0 \tag{3.8}$$

Let $T_0 = 2\pi/\omega_0$ be the resonant period of the oscillator, then

$$T_0 . \Delta E = T_0 E_0 = \pi . K \tag{3.9}$$

Since the damping is very low, T_0 may be considered as the time necessary to measure the average energy E_0, i.e. ΔE.

It is of interest to consider the dependence on the oscillator parameters of the average energy absorbed per unit time by the oscillator. Rewrite (2.5) as

$$e\langle\!\langle \dot{x} . \mathscr{E}(t) \rangle\!\rangle = m\tau \langle\!\langle \ddot{x}^2 \rangle\!\rangle \tag{3.10}$$

but

$$m\tau \langle\!\langle \ddot{x}^2 \rangle\!\rangle = m\tau\omega_0{}^4 \langle\!\langle x^2 \rangle\!\rangle = \tau\omega_0{}^2 . m\omega_0{}^2 \langle\!\langle x^2 \rangle\!\rangle = \tau\omega_0{}^2 . k \langle\!\langle x^2 \rangle\!\rangle$$

and, since the oscillator is very slightly damped, $k . \langle\!\langle x^2 \rangle\!\rangle = K\omega_0/2$. Then the energy absorbed by the oscillator per unit time is:

$$e . \langle\!\langle \dot{x} . \mathscr{E} \rangle\!\rangle = \tau\omega_0{}^2 \frac{K\omega_0}{2} = \tau\omega_0{}^2 E_0 \tag{3.11}$$

where $\tau\omega_0{}^2$ is four times the bandwidth of the oscillator (Surdin, 1970).

To an increase in T_0 corresponds a decrease in E_0 and, due to the narrowing of the bandwidth, even a faster decrease in the average energy absorbed per unit time by the oscillator.

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